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# A Potts/Ising correspondence on thin graphs

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**Abstract.** We note that it is possible to construct a bond vertex model that displays $q$-state Potts criticality on an ensemble of $\phi^3$ random graphs of arbitrary topology, which we denote as 'thin' random graphs in contrast to the fat graphs of the planar diagram expansion.

Since the four-vertex model in question also serves to describe the critical behaviour of the Ising model in field, the formulation reveals an isomorphism between the Potts and Ising models on thin random graphs. On planar graphs a similar correspondence is present only for $q = 1$, the value associated with percolation.

## 1. Introduction

A description of the Potts model on planar random graphs as an ice-type vertex model on the associated medial graphs was developed some years ago by Baxter *et al* [1]. In this paper we formulate a bond vertex description of the Potts model on non-planar random graphs, which we call 'thin' graphs to distinguish them from the fat graphs which appear in a planar graph expansion. The study of spin and vertex models on such thin random graphs is of interest as a way of obtaining mean-field-like exponents [2–5]. Mean-field exponents are obtained because the thin graphs look locally like a Bethe lattice, but all the branches of the tree-like Bethe lattice structure are closed by predominantly large loops†. Planar graphs on the other hand have a loop distribution that is strongly peaked on short loops and a fractal-like baby universe structure [6].

Nonetheless, the methods used for calculating the partition functions of spin models on thin graphs may still be borrowed from the study of the planar random graphs [7]. These methods are based on the observation that planar graphs can be thought of as arising as Feynman diagrams in the perturbative expansion of integrals over $N \times N$ Hermitian matrices. Each edge in such a Feynman diagram is 'fat' or ribbon-like, since it carries two matrix indices. In the $N \to \infty$ limit, fat graphs of planar topology are picked out, since there is an overall factor of $N^{\chi}$ for any graph, where $\chi$ is the Euler characteristic. The $N \to 1$ limit, on the other hand, weights all topologies equally. In this case the matrix fat-graph propagators degenerate to scalars, so we denote the generic random graphs of the $N \to 1$ limit as 'thin' graphs. The Feynman diagram approach used in studying the statistical mechanics of models on planar random graphs‡ still applies for the thin graph case too. In fact, life is even easier in this case since one is dealing with scalar rather than matrix integrals.

† The critical exponents obtained in Bethe lattice calculations are those of mean-field theory, but non-universal quantities such as the critical temperature depend on the coordination number of the Bethe lattice. As one takes the coordination number of the Bethe lattice to infinity one obtains the 'real' mean-field critical temperature.

‡ Which is equivalent to coupling the models to 2D quantum gravity.

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The partition function for a statistical mechanical model on an ensemble of thin random graphs with $2m$ vertices can be defined by means of an integral of the general form [2]

$$Z_m \times N_m = \frac{1}{2\pi \mathrm{i}} \oint \frac{\mathrm{d}\lambda}{\lambda^{2m+1}} \int \frac{\prod_i \mathrm{d}\phi_i}{2\pi \sqrt{\det K}} \exp(-S) \tag{1}$$

where the contour integral over the vertex coupling $\lambda$ picks out graphs with $2m$ vertices, $S$ is an appropriate action, $K$ is the inverse of the quadratic form in this action, and the $\phi_i$ are sufficient variables to describe the matter in the theory. The factor $N_m$ gives the number of undecorated (without matter) graphs in the class of interest and disentangles this usually factorial growth from any phase transitions. For the class of thin $\phi^3$ (three-regular) random graphs we discuss here

$$N_m = \left(\frac{1}{6}\right)^{2m} \frac{(6m-1)!!}{(2m)!!}. \tag{2}$$

In the large $m$, thermodynamic limit, saddle point methods may be used to evaluate equation (1). Similar calculations are familiar in the context of estimations of large-order (i.e. large $m$) behaviour in perturbation theory for quantum mechanical [8] and field theoretical path integrals [9, 10] and vacuum decay effects [11]. The use of simple, scalar integrals and saddle point methods to count classes of (decorated) random graphs at large orders was developed in [3], although such ideas are already latent in the work of [8].

The saddle point equation for $\lambda$ may be decoupled by scaling it out of the action as an overall factor, leaving any critical behaviour residing in the saddle point equations for the matter fields $\phi_i$. Phase transitions appear as an exchange of the dominant saddle point.

## 2. The Potts model on thin graphs

The Hamiltonian for a $q$-state Potts model can be written

$$H = \beta \sum_{\langle ij \rangle} \delta_{\sigma_i,\sigma_j} \tag{3}$$

where the spins $\sigma_i$ take on $q$ values. The action which generates the correct Boltzmann weights for the $q$-state Potts model on $\phi^3$ graphs to be used in equation (1) is [5, 12]

$$S = \frac{1}{2} \sum_{i=1}^{q} \phi_i^2 - c \sum_{i<j} \phi_i \phi_j - \frac{\lambda}{3} \sum_{i=1}^{q} \phi_i^3 \tag{4}$$

where $c$ is $1/(\exp(2\beta)+q-2)$. Since for any $q$ one finds a high-temperature, disordered-phase solution of the form $\phi_i = 1-(q-1)c$, $\forall$i bifurcating to a broken symmetry, ordered-phase solution $\phi_2 = \ldots \phi_{q-1} \neq \phi_1$ at $c = 1/(2q-1)$, an effective action with only two variables $\phi, \tilde{\phi}$, where $\tilde{\phi} = \phi_1, \quad \phi = \phi_2, \phi_3, \ldots, \phi_q$, suffices to describe the transition [5]

$$S_1 = \frac{1}{2}(q-1)[1-c(q-2)]\phi^2 - \frac{\lambda}{3}(q-1)\phi^3 + \frac{1}{2}\tilde{\phi}^2 - \frac{\lambda}{3}\tilde{\phi}^3 - c(q-1)\phi\tilde{\phi}. \tag{5}$$

In the high-temperature phase $\phi = \tilde{\phi}$ and this collapses to

$$S_2 = \frac{q}{2}(1-c(q-1))\phi^2 - \frac{\lambda q}{3}\phi^3. \tag{6}$$

The bifurcation point is *not* the first-order transition point displayed by the model for $q > 2$, but rather a spinodal point. The first-order transition is pinpointed by observing that

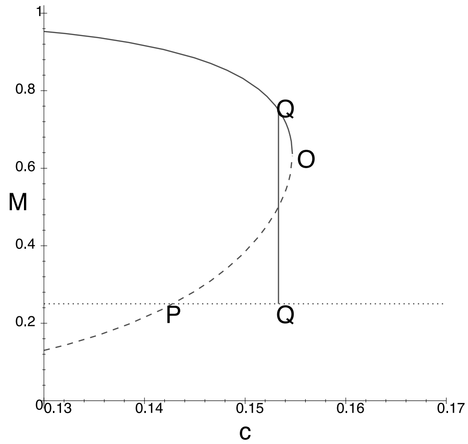


**Figure 1.** The magnetization $m$ for a four-state Potts model as calculated from the saddle point solutions. The high-temperature solution is shown as a dotted line, one low-temperature solution as a solid curve and the other as a dashed curve. The first-order jump in $m$ is at $\boldsymbol{Q}$.

the free energy in the saddle point approximation is the logarithm of the action $S$, so the first-order transition point is given by the $c$ value, and hence temperature satisfying $S_1 = S_2$. This gives the critical value of $c$ as

$$c = \frac{1 - (q-1)^{-1/3}}{q-2} \tag{7}$$

in agreement with other mean-field approaches [13, 14]. This can be seen explicitly by translating between the parameters $\theta = \exp(-2\beta)$ of [14]† and the $c = 1/(\theta^{-1} + q - 2)$ of this paper. The critical value of $\theta$ derived from equation (7)

$$\theta_c = \frac{(q-1)^{1/3} - 1}{q-2} \tag{8}$$

agrees with that presented in [14] for a three coordinated Bethe lattice.

The nature of the phase diagram can best be clarified by examining a diagram of the magnetization defined by

$$M = \frac{\tilde{\phi}^3}{(\tilde{\phi}^3 + (q-1)\phi^3)} \tag{9}$$

versus $c$, as plotted in figure 1 for $q = 4$. We can see that the bifurcation point at $\boldsymbol{P}$, $c = 1/(2q-1)$ for general $q$, lies below the first-order transition point at $\boldsymbol{Q}$, $c = [1-(q-1)^{-1/3}]/(q-2)$. The second spinodal point at $\boldsymbol{O}$, $c = [q - 1 - 2\sqrt{q-1}]/[(q-1)(q-5)]$, is given by the vanishing of a square root in the saddle point solution. As $q \to 2$, $\boldsymbol{O}$, $\boldsymbol{P}$, $\boldsymbol{Q}$ coalesce, the skewed pitchfork of figure 1 becomes symmetrical and we recover the continuous mean-field transition of the Ising model—which is equivalent to the $q = 2$ Potts model. This is to be expected since setting $q = 2$ directly in the action of equation (5) and the magnetization of equation (9) recovers the Ising action and magnetization.

Since $q$ appears explicitly as a parameter in equation (5) the formalism is ideally suited to studying percolation on thin graphs as the $q \to 1$ limit of the Potts model, as well as the random resistor, $q \to 0$, and dilute spin glass, $q \to \frac{1}{2}$, problems.

† Allowing for a factor of two in the respective definitions of $\beta$ in [14] and that used here.

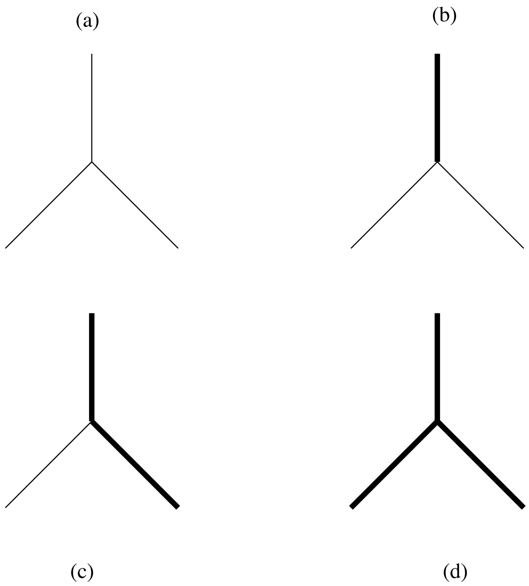


**Figure 2.** The possible bond vertices which appear in the model. The Potts weights on the random graph, which can be read off from equation (11), are: $a = (1+v)/2$ , $b = (\kappa^*)^{1/2}(1-v)/2$, $c = \kappa^*(1+v)/2$ and $d = (\kappa^*)^{3/2}(1-v)/2$.

## 3. A Potts vertex model

In order to recast the Potts action as a bond vertex model we first carry out the following rescaling on equation (5):

$$\phi \to \frac{1}{\sqrt{(q-1)(1-c(q-2))}}\phi \tag{10}$$

which gives both the quadratic terms the canonical coefficient of $\frac{1}{2}$, followed by the linear transformation $\phi \to (X-Y)/\sqrt{2}$, $\tilde{\phi} \to (X+Y)/\sqrt{2}$ and a further rescaling of the quadratic terms and $\lambda$ to obtain

$$S = \frac{1}{2}(X^2+Y^2) - \frac{\lambda}{3}\frac{(1+v)}{2}[X^3 + 3\kappa^* XY^2] - \frac{\lambda(\kappa^*)^{3/2}}{3}\frac{(1-v)}{2}\left[Y^3 + \frac{3}{\kappa^*}YX^2\right] \tag{11}$$

where $\kappa^* = (1-\kappa)/(1+\kappa)$ and

$$\begin{aligned} v &= \frac{1}{(q-1)^{1/2}(1-(q-2)c)^{3/2}} \\ \kappa &= \sqrt{\frac{c^2(q-1)}{1-(q-2)c}}. \end{aligned} \tag{12}$$

The notation in equation (11) has been chosen to facilitate comparison with the Ising model in field below.

We can see from the above that the action of equation (5) for the Potts model on thin $\phi^3$ graphs can be transformed via some rescalings and a linear transformation of the variables into a four-vertex model on the $\phi^3$ graphs. What is more, the vertex model is symmetric since the weight depends only on the number of $X$ and $Y$ bonds at a vertex†. The different vertices in the model are shown in figure 2. Although the result of a straightforward transformation,

† On random graphs the notion of orientation is lost, so there are fewer possibilities for defining independent vertex weights than on regular lattices. On planar random graphs one can still define a cyclic ordering of bonds round a vertex consistently, but even this is lost on thin graphs.

equation (11) entails a surprising consequence. The action for the Ising model (i.e. the $q = 2$ state Potts model) in field on thin graphs† is given by [15]

$$S = \mathrm{Tr}\left\{\frac{1}{2}(X^2 + Y^2) - gXY + \frac{\lambda}{3}[\mathrm{e}^h X^3 + \mathrm{e}^{-h} Y^3]\right\} \tag{13}$$

where $g = \exp(-2\beta)$ and $h$ is the external field. Carrying out the transformations

$$\begin{aligned} X &\to (X + Y)/\sqrt{2} \\ Y &\to (X - Y)/\sqrt{2} \end{aligned} \tag{14}$$

followed by the rescalings $X \to X/(1-g)^{1/2}$, $Y \to Y/(1+g)^{1/2}$, $\lambda \to \sqrt{2}\lambda(1-g)^{3/2}$ again gives a four-vertex model [16]

$$S = \frac{1}{2}(X^2 + Y^2) - \frac{\lambda\cosh(h)}{3}[X^3 + 3g^* XY^2] - \frac{\lambda\sinh(h)(g^*)^{3/2}}{3}\left[Y^3 + \frac{3}{g^*}X^2 Y\right] \tag{15}$$

where $g^* = (1-g)/(1+g)$.

We thus find that the vertex model action for the $q$ state Potts model and that for the Ising model in field are isomorphic under the identifications

$$\begin{aligned} \tanh(h) &= \frac{1-v}{1+v} \\ g &= \kappa. \end{aligned} \tag{16}$$

In fact, backtracking to equation (5) we can see that the rescaling of equation (10) transforms the action into that of an Ising model in field, even before the transformation to a vertex model

$$S \to \frac{1}{2}(\phi^2 + \tilde{\phi}^2) - \kappa\phi\tilde{\phi} - \frac{\lambda v}{3}\phi^3 - \frac{\lambda}{3}\tilde{\phi}^3. \tag{17}$$

The equivalence between the Potts and Ising models is surprising from the physical point of view since we already know that the Potts models display a first-order transition for $q > 2$ [5, 13] whereas the Ising model displays a (mean-field) second-order transition. Things become clearer when we consider the mapping of the Potts values for $c(q, \beta)$ at the critical and spinodal points onto the parameters $\kappa, v$, which play the role of temperature and field in the Ising model (or more precisely $\exp(-2\beta)$ and $\exp(2h)$). The corresponding values of $\kappa$ for $\boldsymbol{O}, \boldsymbol{P}, \boldsymbol{Q}$ are shown in figure 3 and for $v$ in figure 4. We find that the first-order transition point in the Potts models at $\boldsymbol{Q}$, where $c = [1 - (q-1)^{-1/3}]/(q-2)$, maps onto

$$\begin{aligned} \kappa &= \pm\frac{(q-1)^{2/3} - (q-1)^{1/3}}{q-2} \\ v &= 1 \end{aligned} \tag{18}$$

where the sign on the right-hand side of the expression for $\kappa$ is chosen to give a positive answer depending on whether $q <$ or $> 2$. Remarkably, we see that this corresponds to a zero-field point in the Ising model, with $\kappa < \kappa_c = \frac{1}{3}$, the Ising critical value. Since this means that $\beta > \beta_{\text{critical Ising}}$, this point lies on the zero-field line separating the two possible spin orientations in the ordered phase, so the first-order temperature driven transition of the Potts model is mapped onto the field driven transition of the Ising model. As $q \to 2$ from above or below, $\kappa \to \frac{1}{3}$, the Ising critical value, and the transition becomes the continuous mean-field Ising transition.

† And on planar graphs too, if we take $X, Y$ to be matrices.

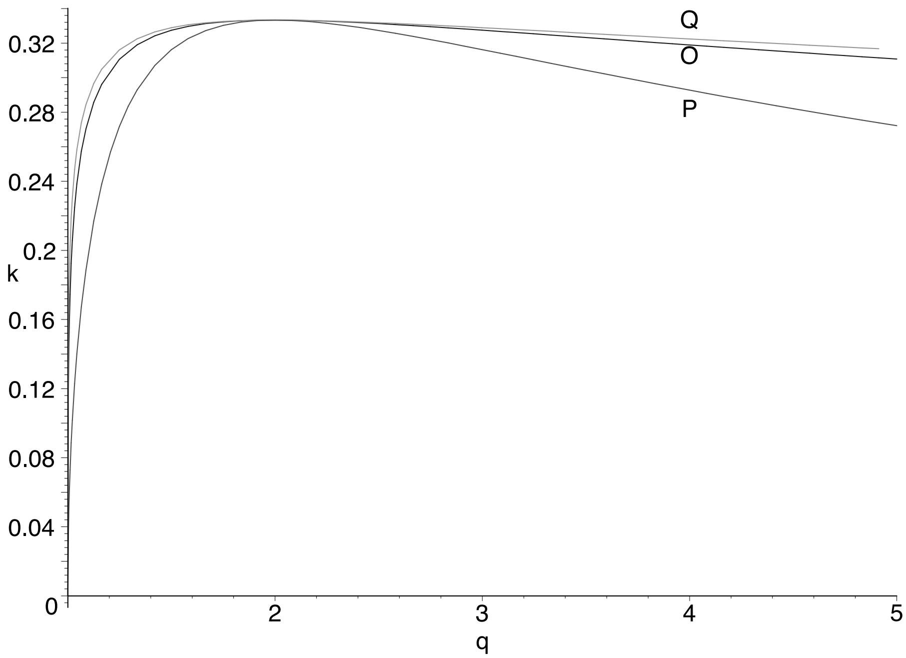


**Figure 3.** $\kappa$ versus $q$ for the points $\boldsymbol{O}$, $\boldsymbol{P}$, $\boldsymbol{Q}$. The curves touch at $q = 2$, $\kappa = \frac{1}{3}$ which is the Ising critical point.

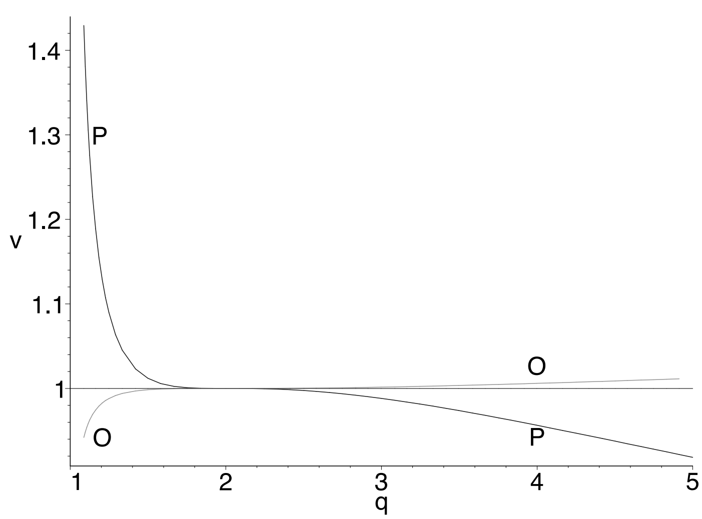


**Figure 4.** $v$ versus $q$ for the points $\boldsymbol{O}$, $\boldsymbol{P}$, $\boldsymbol{Q}$. Since $v = 1$ corresponds to zero field in the equivalent Ising model, we see that $\boldsymbol{Q}$ is a zero-field point for all $q$. For $\boldsymbol{O}$, $\boldsymbol{P}$ on the other hand, $v = 1$ only at $q = 2$.

The spinodal point at $\boldsymbol{P}$ where $c = 1/(2q-1)$ maps onto

$$\kappa = \sqrt{\frac{q-1}{(2q-1)(q+1)}} \qquad v = \frac{2q-1}{(q-1)^{1/2}(q+1)} \tag{19}$$

which tends towards the standard mean-field Ising transition point in zero field as $q \to 2$. Similarly, the other spinodal point at $\boldsymbol{O}$, $c = [q-1-2\sqrt{q-1}]/[(q-1)(q-5)]$, maps onto

$$\kappa = \frac{(q-1)^{1/4}}{(2\sqrt{q-1}+1)^{1/2}(\sqrt{q-1}+2)^{1/2}} \qquad v = \frac{(\sqrt{q-1}+2)^{3/2}(q-1)^{1/4}}{(2\sqrt{q-1}+1)^{3/2}} \tag{20}$$

so we see that both these points are not, in general, zero-field points in the Ising transcription except at $q = 2$.

A natural question to ask is what is so special about $\boldsymbol{Q}$ from the Ising point of view. If we look at the effect of the scaling of equation (10) on the original Potts action of equation (5) and demand a $c$ value which gives $v = 1$ (i.e. zero field) we get precisely $c = [1-(q-1)^{-1/3}]/(q-2)$. The first-order transition point at $c(\boldsymbol{Q})$ is thus the only point which maps onto zero field in the Ising model for general $q$.

## 4. Discussion

The main conclusion of this paper may be simply stated: the Ising model in field and the Potts model on thin $\phi^3$ random graphs may be mapped onto one another. This can be seen either by mapping them both to a four-vertex model, or by directly rescaling the Potts action. The equivalence is, at root, due to the fact that the Potts action of equation (5) requires only two variables, just like the Ising model, even though the Potts spins have $q$ states. We have also seen that the first-order nature of the Potts transitions for $q \neq 2$ and the continuous transition of the Ising model are not in contradiction, since the Potts transition maps onto the field driven transition of the Ising model. The first-order Potts transition point lies on the zero-field Ising locus at $\beta > \beta_{\text{critical Ising}}$ and moves to the continuous Ising transition point as $q \to 2$.

One might enquire as to whether similar relations could exist on planar random graphs where the scalars in actions such as equation (4) are replaced by $N \times N$ Hermitian matrices. Unfortunately, a two variable (in this case matrix variable) effective action such as that in equation (5) does not appear to exist in such a case. However, one can arrive at a $3q+1$ vertex model for planar graphs by using Kazakov's approach [12] of introducing an auxiliary matrix to decouple the Potts interactions. This transforms the matrix action for the $q$ state Potts model:

$$S = \mathrm{Tr}\left\{\frac{1}{2}\sum_{i=1}^{q}\phi_i^2 - c\sum_{i<j}\phi_i\phi_j - \frac{\lambda}{3}\sum_{i=1}^{q}\phi_i^3\right\} \tag{21}$$

to

$$S = \mathrm{Tr}\left\{\frac{1}{2}X^2 + \frac{1+c}{2}\sum_{i=1}^{q}\phi_i^2 - \sqrt{c}\sum_{i}X\phi_i - \frac{\lambda}{3}\sum_{i=1}^{q}\phi_i^3\right\} \tag{22}$$

which after a shift in $\phi_i$ and a rescaling may be written as

$$S = \mathrm{Tr}\left\{\frac{1}{2}X^2 + \sum_{i=1}^{q}\frac{1}{2}\phi_i^2 - \frac{\lambda}{3}\sum_{i=1}^{q}(X^3 + 3v^{1/2}X^2\phi_i + 3vX\phi_i^2 + v^{3/2}\phi_i^3)\right\} \tag{23}$$

where $v = \exp(\beta) - 1$. In general, this will not be equivalent to an Ising model since we have too many matrices, but when $q = 1$, which is related to the problem of percolation, we do recover an Ising-like action.

It would be of some interest to see whether this transcription might shed some light on either percolation or the Ising model in field on planar random graphs. It would also be interesting to explore the relation between the planar graph Potts vertex model of equation (23) and the medial graph vertex model of Baxter *et al* [1].

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